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Solution of the Bernstein Problem in the Non-regular Case

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This paper describes explicitly all non-regular non-degenerate simplicial stochastic Bernstein algebras. Consequently, the Bernstein problem (S. N. Bernstein, *Science Ukraine* **1** (1992), 14–19) in the non-degenerate case is settled, since the regular and exceptional cases have already been examined by Y. Lyubich in the 1970s. Notice that from this result it is possible to explicitly describe every non-regular simplicial algebra (A, Δ) since the simplicial subalgebra $(\langle \mathrm{supp}(A^2)\rangle, [\mathrm{supp}(A^2)])$ is non-degenerate. Also we prove the relevant Lyubich's conjecture (1992, Yu I. Lyubich, *Biomathematics* **22**, 232) in an affirmative way: all normal simplicial stochastic Bernstein algebras are regular. © 2000 Academic Press

Key Words: non-associative algebra; Bernstein algebra; simplicial stochastic algebra.

1. INTRODUCTION

Simplicial stochastic Bernstein algebras were introduced by Lyubich (see also P. Holgate [10]) as algebras representing populations that reach the equilibrium after a generation. In this way Bernstein algebras emerge in connection with the problem in mathematical heredity posed by S. N. Bernstein [1] in 1922.

Non-associative algebras appear in genetics via gametic, zygotic, or copular algebras in a quite natural way when we try to express as a symbolic product the way in which biological characteristics are passed down through generations, and they are in general commutative and have a non-zero homomorphism of the algebra into the field. In a recent survey [20] a general introduction to algebras related to genetics can be found.

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Bernstein proposed the description of all biological mechanisms that follow the Stationary Principle. Let us, following [19], translate to an algebraic language the above-mentioned problem. We say that a commutative algebra A over the field $\mathbb{R}$ has *stochastic realization* if it admits a basis $\Phi = \{e_1, \ldots, e_n\}$, called *stochastic*, such that $\Delta := \{x = \sum_{i=1}^n x_i e_i \colon x_i \geq 0, \sum_{i=1}^n x_i = 1\}$ the simplex spanned by this basis is invariant with respect to the multiplication in the sense $\Delta \cdot \Delta \subset \Delta$. The pair (A, Δ) is called a *simplicial stochastic algebra* (abbreviated as *simplicial algebra*). The mapping $\omega \colon A \to \mathbb{R}$ given by $\omega(\sum_{i=1}^n x_i e_i) = \sum_{i=1}^n x_i$ is a non-zero homomorphism. A simplicial algebra (A, Δ) is called *Bernstein*, if for all $x \in A$,

$$(x^2)^2 = \omega(x)^2 x^2. \tag{1}$$

The Bernstein Problem is equivalent to explicitly describing all simplicial Bernstein algebras. The cases $n = 1, 2$ are trivial. Bernstein solved the problem for $n = 3$ and some particular cases for $n = 4$. The solution, for $n > 3$, in the *regular* and *exceptional* cases was found by Lyubich in the 1970s (see [11−15, 18]). In view of the fact that for $n \leq 4$ every Bernstein algebra is regular or exceptional, the problem for $n = 4$ was completely solved by Lyubich. The Bernstein problem in the non-regular case for $n = 5$ and $n = 6$ was solved in [5] and [8], respectively. Finally, the non-regular cases for types $(n-2, 2)$ and $(3, n-3)$ were obtained by the author in [7] and [9], respectively. Here we solve, for all n, the Bernstein problem in the non-degenerate non-regular case. Thus, this work and the above-cited papers settle the Bernstein problem in the non-degenerate case. We note that every simplicial Bernstein algebra (A, Δ) can be obtained from its non-degenerate simplicial algebra $(\langle \mathrm{supp}(A^2) \rangle, [\mathrm{supp}(A^2)])$, and hence all simplicial Bernstein algebra can be explicitly described.

The following well-known facts about Bernstein algebras will be used in this paper (see [4, 10, 16, 18, 21]). A Bernstein algebra has a unique weight homomorphism ω. The ideal $\ker \omega$ of co-dimension 1 is called the *barideal* of A, $\mathrm{bar}(A)$. Every Bernstein algebra contains at least one non-zero idempotent element. If e is an idempotent element, then A has a Peirce decomposition $A = \mathbb{R}e \oplus U_e \oplus Z_e$, where $\mathrm{bar}(A) = U_e \oplus Z_e$ and

$$U_e = \{x \in A \colon 2ex = x\}, \qquad Z_e = \{x \in A \colon ex = 0\}. \tag{2}$$

Products between elements of U_e and Z_e satisfy the relations

$$U_e^2 \subset Z_e, \qquad U_e Z_e \subset U_e, \qquad Z_e^2 \subset U_e, \tag{3}$$

$$u^3 = 0 = u(uz) = uz^2 = (uz)^2 = (u^2)^2 \tag{4}$$

for all $u \in U_e$ and $z \in Z_e$. The subspace

$$U_0 := U_e \cap \operatorname{ann}(U_e) = \{u \in U_e \colon uU_e = (0)\}.$$

is an ideal of A independent of the idempotent e considered. Furthermore, the following relations hold (for all $u \in U_e$ and $z \in Z_e$):

$$(uz)z, z^2 \in U_0. \tag{5}$$

Therefore, if $uz \in \mathbb{R}u$, then $uz \in U_0$. Notice that $U_0(U_e \oplus U_e^2) = (0)$ by (3) and (4). The set of idempotent elements $I(A)$ of A is given by

$$I(A) = \{x^2 \colon \omega(x) = 1\} = \{e + u + u^2 \colon u \in U_e\},$$

and if $\bar{e} = e + \bar{u} + \bar{u}^2$ is another idempotent, then

$$U_{\bar{e}} = \{u + 2u\bar{u} \colon u \in U_e\}, \qquad Z_{\bar{e}} = \{z - 2(\bar{u} + \bar{u}^2)z \colon z \in Z_e\}. \tag{6}$$

Dimensions of U_e, Z_e, U_e^2, and $U_eZ_e + Z_e^2$ do not depend on the choice of the idempotent element, which justifies the following definitions: the *type* of A as the pair (m, δ), where $m - 1 = \dim U_e$ and $\delta = \dim Z_e$ (so $n = \dim A = m + \delta$), and m is called the *rank* of A, $m = \operatorname{rk} A$; a Bernstein algebra in which $U_eZ_e = (0)$ and $Z_e^2 = (0)$ is called *regular* and a Bernstein algebra in which $U_e^2 = (0)$ is said to be *exceptional*. The algebra A is called *nuclear* if $A^2 = A$. Notice that if A is a Bernstein algebra, then $A^2 = \mathbb{R}e \oplus U_e \oplus U_e^2$ for all $e \in I(A)$ and hence A^2 is nuclear.

In the following (A, Δ) will be a simplicial Bernstein algebra of type (m, δ) with $\Phi = \{e_i\}_{i=1}^n$ as a stochastic basis such that $\Delta = [\Phi]$. The simplicial algebra (A, Δ) is called *degenerate* if there exists j such that A^2 is a subspace of $\ker(e_j^*)$. In other cases the simplicial algebra is called *non-degenerate*. Furthermore a non-degenerate simplicial algebra (A, Δ) is called *normal* if it is *externally* and *internally irreducible*, where externally irreducible means that for every pair j, k $(j \neq k)$ and for every scalar $\alpha > 0$ we have that $A^2 \not\subset \ker(e_j^* + \alpha e_k^*)$, and internal irreducible means that there is no pair j, k, $(j \neq k)$ such that $e_j - e_k \in \operatorname{ann}(A)$. Here $\{e_i^*\}$ is the dual basis of $\{e_i\}$ and $\operatorname{ann}(A) := \{x \in A \colon xA = 0\}$.

For a finite sequence $\phi = \{a_1, a_2, \dots, a_r\}$ of elements of A, $\langle a_1, a_2, \dots, a_r \rangle$ denotes the linear span of ϕ and $[a_1, a_2, \dots, a_r]$ the convex hull of ϕ.

If $x = \sum_i x_i e_i$, then the *support* of x with respect to the basic Φ, denoted by $\operatorname{supp}(x)$, is the set of basic vectors for which the coefficient is non-zero, that is, $e_i \in \operatorname{supp}(x)$ if and only if $x_i \neq 0$. Finally, if X is a subset of A we define $\operatorname{supp}(X) = \bigcup\{\operatorname{supp}(x) \colon x \in X\}$. Furthermore, if $\Psi = \{a_i\}$ is another basis and $x = \sum_k y_k a_k \in A$, then the support of x with respect to Ψ, denoted by $\operatorname{supp}_\Psi(x)$, is the set $\{a_k \in \Psi \colon y_k \neq 0\}$.

A *face* of the simplex Δ is every subset $[e_{i_1}, \dots, e_{i_m}]$, where $e_{i_1}, \dots, e_{i_m}$ are basic vectors. For every face Γ, we denote by $\operatorname{Int}(\Gamma)$ the interior of the face Γ with respect to its affine hull. Notice that $x \in \Delta$ belongs to $\operatorname{Int}(\Gamma)$ if and only if $\Gamma = [\operatorname{supp}(x)]$.

We say that a simplicial algebra (B, Γ) is a *simplicial algebra* of (A, Δ) if B is a subalgebra of A and $\Gamma = B \cap \Delta$. We know that for every non-degenerate simplicial subalgebra (B, Γ) or (A, Δ), the face Γ is *essential* in the sense $\operatorname{Int}(\Gamma) \cap I(A) \neq \emptyset$. In this case we say that Γ is $(k-1)$-essential, where k is the rank of B. Conversely, if a face Γ of Δ is essential, then $B := \langle \Gamma \rangle$ is a subalgebra of A and (B, Γ) is a non-degenerate simplicial subalgebra of (A, Δ).

With respect to simplicial subalgebras we will use a result proved in [17, 18] in a topological context. It can be reformulated in an algebraic way as

LEMMA 1.1. *Every simplicial Bernstein algebra of rank m has at least m non-degenerate simplicial subalgebras of rank* 1. *Besides, if (B_1, Γ_1) and (B_2, Γ_2) are two different non-degenerate subalgebras of (A, Δ) with rank* 1, *then $B_1 \cap B_2 = \{0\}$.*

The following is central in our analysis:

THEOREM 1.1 (Theorem 2.1 of [6]). *Let (A, Δ) be a simplicial Bernstein algebra of rank m, and with $\Phi = \{e_i\}_{i=1}^n$ as a stochastic basis, $\Delta = [\Phi]$. Then there are an idempotent element e, a basis $u_2, \dots, u_m$ of U_e, and elements $z_i \in Z_e$ such that*

$$e_i = e + \sum_{t=2}^{m} \mu_{it} u_t + z_i \qquad (1 \leq i \leq n),$$

where $\mu_{it} \geq 0$ and $e + \lambda u_k \in \Delta$ if and only if $0 \leq \lambda \leq 1$ for all t and k. Furthermore, if the stochastic basis is non-degenerate, then for every i there are at most two non-zero coefficients μ_{it}.

2. NON-DEGENERATE SIMPLICIAL BERNSTEIN ALGEBRA

In the following we will assume that the simplicial Bernstein algebra (A, Δ) is non-degenerate of rank m. Denote by u_1 the null vector. In the proof of the main theorem we will use some notations and results that follow from Theorem 2.1 of [6]. According to [6] we know the existence of a partition $\{\Omega_l\}_{l=1}^{m+\gamma}$ of the stochastic basis, an idempotent e in Δ, a basis $\{u_2, u_3, \dots, u_m\}$ of U_e, and a basis $\{z_{lt_l}\colon 1 \leq l \leq m+\gamma,\ 1 \leq t_l \leq n_l\}$ of Z_e

such that the expression of the stochastic basis Φ, after reindexing, with respect to

$$\Psi = \left\{e, u_i, z_{lt_l}\colon 2 \le i \le m, 1 \le l \le m + \gamma, 1 \le t_l \le n_l\right\}$$

is of the form (for $2 \le i \le m$ and $1 \le j \le \gamma$)

$$\Phi_1 = \begin{cases} e_{10} = e + \nu_{10} u_1 + \sum_{k=1}^{r_1} \alpha_{1k} z_{1k} & \\ e_{1t} = e + \nu_{1t} u_1 + z_{1t} & (1 \le t \le n_1) \end{cases} \tag{7}$$

$$\Phi_i = \begin{cases} e_{i0} = e + \nu_{i0} u_i + z_{i0} + \sum_{k=1}^{r_i} \alpha_{ik} z_{ik} & \\ e_{it} = e + \nu_{it} u_i + z_{it} & (1 \le t \le n_i) \end{cases} \tag{8}$$

$$\Phi_{m+j} = \left\{ e_{m+jt} = e + \lambda_{jt} u_{i_j} + \overline{\lambda_{jt}} u_{k_j} + z_{m+jt} \qquad \left(1 \le t \le n_{m+j}\right), \right. \tag{9}$$

where $z_{i0} \in \langle z_{1k}\colon 1 \le k \le n_1 \rangle$; $\alpha_{it} < 0$; $0 < \nu_{it} \le 2$; $0 < \lambda_{jt}, \overline{\lambda_{jt}}$; $2 < i_j < k_j \le m$; and all the pairs (i_j, k_j) are distinct. Furthermore,

$$e + \lambda u_j \in \Delta \Leftrightarrow 0 \le \lambda \le 1. \tag{10}$$

As suggested by the previous presentation of the stochastic basis, we define, following [6], the subspaces (for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, \gamma$)

$$W_i := Z_e \cap \left\langle \operatorname{supp}_\Psi(\operatorname{supp}(e + u_i)) \right\rangle, \qquad W_{m+j} = \{0\}$$

$$W := W_1 + W_2 + \cdots + W_{m+\gamma},$$

$$Y_i := \langle z_{it} \in \Psi\colon z_{it} \notin W \rangle, \qquad Y_{m+j} := \langle z_{m+jt}\colon 1 \le t \le n_{m+j} \rangle$$

$$Y := Y_1 + Y_2 + \cdots + Y_{m+\gamma}$$

$$Z_i := W_i \oplus Y_i, \qquad Z_{m+j} := W_{m+j} \oplus Y_{m+j}.$$

LEMMA 2.1 [6]. *If $x \in \Delta$ and μ_i is the coefficient of u_i in the expression of x with respect to Ψ, then $\mu_i \ge 0$.*

LEMMA 2.2 [6]. *Basic properties of above subspaces are (for $1 \le i \le k \le m$ and $1 \le j \le \gamma$)*

(i) *The vector z_{it} belongs to W_i if and only if $t \le r_i$.*

(ii) *If $i \ne k$, then $Z_i \cap Z_k \subset \langle z_{1t}\colon 1 \le t \le n_1 \rangle$.*

(iii) *Let $a = e + u + z$, $a' = e + u' + z' \in \Delta$. If $z_{it} \in Y$, then*

$$z_{it} \in \operatorname{supp}_\Psi(uu') \Leftrightarrow e_{it} \in \operatorname{supp}(aa');$$

$$z_{m+jt} \in \operatorname{supp}_\Psi(uu') \Leftrightarrow \Omega_{m+j} \subset \operatorname{supp}(aa').$$

If $z_{it} \in W$, $i > 1$, *and* $t > 0$, *then*

$$z_{it} \in \operatorname{supp}_\Psi(uu') \Rightarrow \{e_{i0}, e_{it}\} \cap \operatorname{supp}(aa') \neq \emptyset.$$

(iv) *For each* $z \in \Psi \cap W_i$, *there exist two vectors*, f^- *and* f^+, *in* $[\operatorname{supp}(e + u_i)] \subset [\Omega_1 \cup \Omega_i]$ *with the form*

$$f^- = e + \gamma^- u_i + \varepsilon^- z, \qquad f^+ = e + \gamma^+ u_i + \varepsilon^+ z, \tag{11}$$

where $\gamma^-, \gamma^+ \geq 0$ *and* $\varepsilon^- < 0 < \varepsilon^+$.

(v) Y *is a subspace of* $\langle \operatorname{supp}_\Psi(U_e^2) \rangle$.

(vi) $\langle \operatorname{supp}_\Psi(u_i u_k) \rangle \subset \langle z_{1t} \colon\ 1 \leq t \leq n_1 \rangle + Z_i + Z_k + \sum\{\langle \Omega_{m+j} \rangle \colon (i_j, k_j) = (i, k)\}$.

(vii) *If* $u_i \in U_0$, $i > 1$, *then* $Y_i = \{0\}$.

(viii) *The vector* z_{m+jt} *lies in* $\operatorname{supp}_\Psi(u_i u_k)$ *if and only if* $(i, k) = (i_j, k_j)$. *Furthermore,*

$$u_{i_j} u_{k_j} \notin \left\langle u_i u_k \colon (i, k) \neq (i_j, k_j) \text{ and } 2 \leq i \leq k \leq m \right\rangle.$$

(ix) *Finally,*

$$U_0 := U_e \cap \operatorname{ann}(U_e) \subset \left\langle u_i \colon i \notin \cup_{j=1}^{\gamma} \{i_j, k_j\} \right\rangle.$$

LEMMA 2.3 [6]. *The following relations hold*:

$$U_e^3 = (0), \qquad (\mathbb{R}u_i) \cdot \left\langle \operatorname{supp}_\Psi\left(U_e^2\right) \right\rangle \subset \mathbb{R}u_i, \qquad \left\langle \operatorname{supp}_\Psi\left(U_e^2\right) \right\rangle^2 = (0).$$

LEMMA 2.4. *The relations*

$$Z_i(\mathbb{R}u_k \oplus Z_k) \subset \langle u_i, u_k \rangle \qquad \text{and} \qquad Z_i Z_{m+j} \subset \mathbb{R}u_i$$

hold for all $i, k \in \{1, 2, \dots, m\}$ *and all* $j \in \{1, 2, \dots, \gamma\}$.

Proof. According to Lemma 2.3 it suffices to prove that $W_i(\mathbb{R}u_k)$, $W_i Z_k \subset \langle u_i, u_k \rangle$, and $W_i Z_{m+j} \subset \mathbb{R}u_i$, since $Y_i, Y_k, Y_{m+j} \subset \operatorname{supp}_\Psi(U_e^2)$ by (v) of Lemma 2.2. Let $z \in \Psi \cap W_i$ and consider the vectors f^- and f^+ defined in (11). The elements

$$f^\tau(e + u_k) = e + \left(\frac{\gamma^\tau}{2} u_i + \frac{1}{2} u_k + \varepsilon^\tau z u_k\right) + (\varepsilon^\tau u_i u_k), \qquad \tau \in \{-, +\}$$

belong to Δ, and hence Lemma 2.1 implies that $z u_k \in \langle u_i, u_k \rangle$ since $U_e Z_e \subset U_e$. In view of $W_i = \langle \Psi \cap W_i \rangle$, we conclude that $W_i(\mathbb{R}u_k) \subset \langle u_i, u_k \rangle$. Next, for each $z' \in \Psi \cap Z_k$, there exists an element $a \in \Delta$ of the form $a = e + \nu u_k + z'$. Indeed, if $z' = z_{kt}$, $t > 0$, then we can take as

a the basic vector e_{kt}, and if $z' = z_{1t}$, $t > 0$, then we take as a the basic vector e_{1t}. The elements $f^- a$ and $f^+ a$ lie in Δ,

$$f^\tau a = e + \left(\frac{\gamma^\tau}{2}u_i + \frac{\nu}{2}u_k + \varepsilon^\tau \nu z u_k + \gamma^\tau z' u_i + \varepsilon^\tau z z'\right) + \left(\varepsilon^\tau \nu u_i u_k\right),$$
$$\tau \in \{-,+\},$$

so $zz' \in \langle u_i, u_k\rangle$ and $W_i Z_k \subset \langle u_i, u_k\rangle$, since we have proved above that $Z_i(\mathbb{R}u_k)$, $Z_k(\mathbb{R}u_i) \subset \langle u_i, u_k\rangle$. Finally, for each $z_{m+jt} \in \Psi \cap Z_{m+j}$ we have that $f^- e_{m+jt}$, $f^+ e_{m+jt} \in \Delta$ and hence $zz_{m+jt} \in \langle u_i, u_{i_j}, u_{k_j}\rangle$. Then, because $Z_e^2 \subset U_0$ and by (ix) of Lemma 2.2, $zz_{m+jt} \in U_0 \cap \langle u_i, u_{i_j}, u_{k_j}\rangle \subset \mathbb{R}u_i$. ∎

COROLLARY 2.1. *The following relations hold for all k and j ($2 \le k \le m$; $1 \le j \le \gamma$):*

$$Z_1^2 = (0), \qquad Z_1(\mathbb{R}u_k \oplus Z_k) \subset \mathbb{R}u_k, \qquad Z_1 Z_{m+j} = (0).$$

COROLLARY 2.2. *If $u_i \notin U_0$, then $(\mathbb{R}u_i \oplus Z_i)Z_i = (0)$.*

Proof. If $z \in Z_i$, then by Lemma 2.4 we get that $u_i z \in \mathbb{R}u_i$. So there exists $\alpha \in \mathbb{R}$ such that $u_i z = \alpha u_i$. Then $(u_i z)z = (\alpha u_i)z = \alpha^2 u_i$ belongs to U_0 by (5). Then, because $u_i \notin U_0$, we have $\alpha = 0$ and hence $u_i z = 0$.

Finally, let $z, z' \in Z_i$. We know that in a Bernstein algebra $Z_e^2 \subset U_0$ and hence $zz' \in U_0$. On the other hand, from Lemma 2.4 it follows that $zz' \in \mathbb{R}u_i$. Consequently, $zz' \in U_0 \cap \mathbb{R}u_i = \{0\}$ since $u_i \notin U_0$. ∎

LEMMA 2.5. *The relation $U_e Z_e \subset U_0$ holds.*

Proof. We shall prove that $(\mathbb{R}u_i)Z_e \subset U_0$ for $i = 2, \dots, m$. If $u_i \in U_0$, then the result is trivial since U_0 is an ideal of A. Next, we know that $(\mathbb{R}u_i)\langle \mathrm{supp}_\Psi(U_e^2)\rangle \subset (\mathbb{R}u_i)$ therefore if this product is different from zero, then there exists $z \in Z_e$ with $u_i z = u_i$ and by (5) we have that $u_i = (u_i z)z \in U_0$. We have obtained that $(\mathbb{R}u_i)\langle \mathrm{supp}_\Psi(U_e^2)\rangle \subset (\mathbb{R}u_i) \cap U_0$. Therefore it remains to prove that $(\mathbb{R}u_i)W \subset U_0$. We already know that $(\mathbb{R}u_i)W_i \subset (\mathbb{R}u_i) \cap U_0$. Now we will see that $(\mathbb{R}u_i)W_i \subset U_0$ for $k \ne i$. We know from Lemma 2.4 that $(\mathbb{R}u_i)W_k \subset \langle u_i, u_k\rangle$. Let $u = \alpha u_i + \beta u_k = u_i z$ with $z \in \Psi \cap W_k$. If $\alpha \ne 0$, then $\alpha u_i z + \beta u_k z = (\alpha u_i + \beta u_k)z = (u_i z)z \in U_0$ by (5), and since $u_k z \in U_0$ (we know by Corollary 2.2 that if $u_k z \ne 0$, then $u_k \in U_0$) we conclude that $u_i z \in U_0$. Let us suppose, finally, that $\alpha = 0$ and $u \ne 0$, that is, $u = \beta u_k = u_i z$ with $\beta \ne 0$. Then $u_i u_k = \beta^{-1} u_i(u_i z) = 0$ by the second relation of (4). For every $h \in \{2, \dots, m\}$, $u_h u_k = \beta^{-1} u_h(u_i z) = -\beta^{-1} u_i(u_h z) = 0$, since if $u_h z \notin \mathbb{R}u_k$ then by the previous part $u_h z \in U_0$, and if $u_h z = \lambda u_k$ then $u_i(\lambda u_k) = \lambda u_i u_k = 0$. ∎

COROLLARY 2.3. *If $u_i \notin U_0$ and $u_k \in U_0$ then* $(\mathbb{R}u_i \oplus Z_i)Z_k \subset \mathbb{R}u_k$.

Proof. We know that $(\mathbb{R}u_i)Z_k$, $Z_iZ_k \subset \langle u_i, u_k\rangle \cap U_0$. On the other hand, because $u_i \notin U_0$, we get that $\langle u_i, u_j\rangle \cap U_0 = \mathbb{R}u_k$. ∎

For each i, $2 \le i \le m$, let ν_i be the scalar $\nu_i := \max\{\nu_{it}\colon t = 0, 1, \dots, n_i\}$ and

$$\mathfrak{e}_i := e + \nu_i u_i + \nu_i^2 u_i^2.$$

LEMMA 2.6. *If $u_i \notin U_0$, then $\mathfrak{e}_i \in [e_{it} \in \Omega_i\colon \nu_{it} = \nu_i]$, so $\mathfrak{e}_i \in \Delta$.*

Proof. Assume that $u_i \notin U_0$. There exists $e_{it} \in \Phi$ such that $\nu_{it} = \nu_i$; that is, this basic vector has the form $e_{it} = e + \nu_i u_i + z_{it}$. Then, from Corollary 2.2 it follows that $e_{it}^2 = e + \nu_i u_i + \nu_i^2 u_i^2 = \mathfrak{e}_i$ lies in the basic simplex Δ. Now it is easy to check that $\operatorname{supp}(e_{it}^2) \subset \Omega_0 \cup \Omega_i$,

$$e_{it}^2 = \left(\sum_{k=0}^{n_0} \xi_k e_{0k}\right) + \left(\sum_{k=0}^{n_i} \zeta_k e_{ik}\right),$$

where $\xi_k \ge 0$, $\zeta_k \ge 0$, and $(\sum_{k=0}^{n_0} \xi_k) + (\sum_{k=0}^{n_i} \zeta_k) = 1$. As usual in this paper, $\pi_i(x)$ will denote the coefficient of u_i in the Ψ expression of x. Then

$$\begin{aligned}
\nu_i = \pi_i\big(e_{it}^2\big) &= \pi_i\left(\left(\sum_{k=0}^{n_0} \xi_k e_{0k}\right) + \left(\sum_{k=0}^{n_i} \zeta_k e_{ik}\right)\right) \\
&= \left(\sum_{k=0}^{n_0} \xi_k \pi_i(e_{0k})\right) + \left(\sum_{k=0}^{n_i} \zeta_k \pi_i(e_{ik})\right) = \sum_{k=0}^{n_i} \zeta_k \nu_{ik} \\
&\le \sum_{k=0}^{n_i} \zeta_k \nu_i = \left(\sum_{k=0}^{n_i} \zeta_k\right) \nu_i \le \nu_i.
\end{aligned}$$

Thus, each inequality is equality and hence we have the following: (a) $\xi_k = 0$ for all k, (b) $\sum_{k=0}^{n_i} \zeta_k = 1$, and (c) if $\zeta_k \ne 0$, then $\nu_{ik} = \nu_i$. ∎

COROLLARY 2.4. *If $u_i \notin U_0$ and $u_i^2 = 0$, then*

(i) $z_{i0} = 0$ *and* $\nu_{it} = \nu_i = 1$ *for* $0 \le t \le r_i$; *that is, these vectors have the form*

$$e_{i0} = e + u_i + \sum_{k=1}^{r_i} \alpha_{ik} z_{ik}$$

$$e_{it} = e + u_i + z_{it} \qquad (1 \le t \le r_i).$$

(ii) $\operatorname{supp}(e + u_i) = \{e_{it} \in \Phi\colon 0 \le t \le r_i\}$.

(iii) $Z_i = \langle z_{it}\colon 1 \le t \le n_i\rangle$ *and* $Z_i \cap Z_k = \{0\}$ *for all* $k \ne i$.

Proof. In view of Lemma 2.6 the vector $\mathfrak{e}_i + e + \nu_i u_i \in \Delta$, so (10) forces the inequality $\nu_i \leq 1$. Also by (10) the vector $e + u_i \in \Delta$, so $e + u_i \in [\Omega_0 \cup \Omega_i]$, and this implies that $1 \leq \nu_i$. Consequently, we have proved that $\nu_i = 1$. Finally, from Lemma 2.6 and (ii) of Lemma 2.2 follow the results (i)–(iii). ∎

LEMMA 2.7. *If $u_i, u_k \notin U_0$ and $(\mathbb{R}u_i)Z_k + (\mathbb{R}u_k)Z_i + Z_iZ_k \neq (0)$, then* (a) $Y_i = Y_k = (0)$; (b) $r_i = n_i$ *and* $r_k = n_k$; (c) $\nu_{it} = \nu_i$ *for* $t = 1, \dots, n_i$ *and* $\nu_{kt} = \nu_k$ *for* $t = 1, \dots, n_k$; *that is, these vectors have the form*

$$e_{i0} = e + \nu_i u_i + z_{i0} + \sum_{l=1}^{r_i} \alpha_{il} z_{il} \qquad e_{k0} = e + \nu_k u_k + z_{k0} + \sum_{l=1}^{r_k} \alpha_{kl} z_{kl}$$

$$e_{it} = e + \nu_i u_i + z_{it} \quad (1 \leq t \leq n_i) \qquad e_{kt} = e + \nu_k u_k + z_{kt} \quad (1 \leq t \leq n_k).$$

Furthermore,

$$\mathfrak{e}_i = \frac{1}{1 - \sum_{l=1}^{r_i} \alpha_{il}} \left(e_{i0} - \sum_{l=1}^{r_i} \alpha_{il} e_{il} \right),$$

$$\mathfrak{e}_k = \frac{1}{1 - \sum_{l=1}^{r_k} \alpha_{kl}} \left(e_{k0} - \sum_{l=1}^{r_k} \alpha_{kl} e_{kl} \right) \tag{12}$$

and

$$\Omega_i^2 = \{\mathfrak{e}_i\}, \qquad \mathfrak{e}_i \mathfrak{e}_k = \tfrac{1}{2}(\mathfrak{e}_i + \mathfrak{e}_k), \qquad \Omega_k^2 = \{\mathfrak{e}_k\}.$$

Proof. We first note that $(\mathbb{R}u_i)Z_k + (\mathbb{R}u_k)Z_i + Z_iZ_k = \langle u_i, u_k \rangle \cap U_0$ since we have proved that $U_e Z_e \subset U_0$ for a non-degenerate simplicial Bernstein algebra. Therefore there is $u = \alpha u_i - \beta u_j$ with $\alpha \neq 0$ and $\beta \neq 0$ such that $\langle u_i, u_j \rangle \cap U_0 = \mathbb{R}u$. Thus, for each u_l we have $0 = uu_l = (\alpha u_i - \beta u_k)u_l = \alpha u_i u_l - \beta u_k u_l$, so that $\langle u_i u_l \rangle = \langle u_k u_l \rangle$. Consequently, for $l \in \{i, k\}$, we get that $\langle u_i^2 \rangle = \langle u_i u_k \rangle = \langle u_k^2 \rangle$ and hence $\operatorname{supp}(u_i^2) = \operatorname{supp}(u_k^2) \subset (\langle z_{1t}\colon\ 1 \leq t \leq n_1 \rangle + Z_i) \cap (\langle z_{1t}\colon\ 1 \leq t \leq n_1 \rangle + Z_k) = \langle z_{1t}\colon 1 \leq t \leq n_1 \rangle$ by (ii) of Lemma 2.2. Next, if $l \notin \{i, k\}$, then $\operatorname{supp}(u_i u_l) = \operatorname{supp}(u_k u_l)$, and by (vi) and (ix) of Lemma 2.2 $\operatorname{supp}(u_i u_l)$ is a subspace of $(\langle z_{1t}\colon\ 1 \leq t \leq n_1 \rangle + Z_i + Z_l) \cap (\langle z_{1t}\colon\ 1 \leq t \leq n_1 \rangle + Z_k + Z_l) = \langle z_{1t}\colon\ 1 \leq t \leq n_1 \rangle + Z_l$. Thus, we have proved that $Y_i = Y_k = (0)$, $r_i = n_i$, and $r_k = n_k$.

On the other hand, we know that $\mathfrak{e}_i = e + \nu_i u_i + \nu_i^2 u_i^2 \in [e_{it}\colon \nu_{it} = \nu_i] \subset \Omega_i$ and hence $\mathfrak{e}_i = \sum_{t=0}^{r_i} \xi_t e_{it}$ with $\sum_{t=0}^{r_i} \xi_t = 1$ and $\xi_t \geq 0$. Then, because $u_i^2 \in \langle z_{1t}\colon 1 \leq t \leq n_1 \rangle$, we have that $\alpha_{it}\xi_0 + \xi_t = 0$ for $t = 1, \dots, r_i$. Therefore we have $\langle u_i^2 \rangle = \langle z_{i0} \rangle$ and $\xi_t > 0$ and $\nu_{it} = \nu_i$ for all t. Similar results are obtained for the index k, so (12) is established. Note that we prove that

$$\langle z_{i0}\rangle = \langle u_i^2\rangle = \langle u_i u_k\rangle = \langle u_k^2\rangle = \langle z_{k0}\rangle. \tag{13}$$

Now from Corollary 2.2 we get that $\Omega_i^2 = \{\mathfrak{e}_i\}$ and $\Omega_k^2 = \{\mathfrak{e}_k\}$. Also we already know that A^2 is regular and hence

$$\mathfrak{e}_i \mathfrak{e}_k = e + \tfrac{1}{2}(\nu_i u_i + \nu_k u_k) + \nu_i \nu_k u_i u_k.$$

Since $u_i u_k \in \langle z_{1t}\colon\ 1 \le t \le n_1\rangle$ we have that $\mathfrak{e}_i \mathfrak{e}_k \in [\Omega_0 \cup \{e_{it}\}_{t=0}^{r_i} \cup \{e_{kt}\}_{t=0}^{r_k}]$, so that $\mathfrak{e}_i \mathfrak{e}_k = \sum_{l=0}^{n_0} \xi_l e_{0l} + \sum_{l=0}^{r_i} \zeta_l e_{il} + \sum_{l=0}^{r_k} \varsigma_l e_{kl}$. Then

$$\tfrac{1}{2}\nu_i = \pi_i(\mathfrak{e}_i \mathfrak{e}_k) = \sum_{l=0}^{r_i} \zeta_l \pi_i(e_{il}) = \sum_{l=0}^{r_i} \zeta_l \nu_i = \left(\sum_{l=0}^{r_i} \zeta_l\right)\nu_i,$$

and hence $\sum_{l=0}^{r_i} \zeta_l = 1/2$. In an analogous way we obtain that $\sum_{k=0}^{r_k} \varsigma_l = 1/2$. This forces $\xi_t = 0$ for $t = 0, \ldots, n_0$. Next, because $u_i u_k \in \langle z_{1t}\colon 1 \le t \le n_1\rangle$ we have that $\alpha_{il}\zeta_0 + \zeta_l = 0$ for $l = 1, \ldots, r_i$ and $\alpha_{kl}\varsigma_0 + \varsigma_l = 0$ for $l = 1, \ldots, r_k$, from which it follows that $\mathfrak{e}_i \mathfrak{e}_k = (\mathfrak{e}_i + \mathfrak{e}_k)/2$. ∎

LEMMA 2.8. *If* $u_i, u_k \notin U_0$, *then* $(\mathbb{R}u_i)Z_k, (\mathbb{R}u_k)Z_i,\ Z_i Z_k \subset \mathbb{R}(\nu_i u_i - \nu_k u_k)$.

Proof. If $\langle u_i, u_k\rangle \cap U_0 = \{0\}$, then the result is trivial, since we have proved that $U_e Z_e + Z_e^2 \subset U_0$ and hence $(\mathbb{R}u_i)Z_k + (\mathbb{R}u_k)Z_i + Z_i Z_k \subset \langle u_i, u_k\rangle \cap U_0 = \{0\}$. Thus, we now suppose that $\langle u_i, u_k\rangle \cap U_0 \neq \{0\}$. Then there exists $u = \alpha u_i - \beta u_k$ with $\alpha \neq 0$ and $\beta \neq 0$ such that $\langle u_i, u_k\rangle \cap U_0 = \mathbb{R}u$. We prove the lemma in two steps.

First we assume that $\langle u_i, u_k\rangle^2 \neq (0)$. Since $u \in U_0$ we have that $0 = u_i u = \alpha u_i^2 - \beta u_i u_k$ and $0 = u u_k = \alpha u_i u_k - \beta u_k^2$, so that

$$\frac{\alpha}{\beta} u_i^2 = u_i u_k = \frac{\beta}{\alpha} u_k^2. \tag{14}$$

Next, the relation $\mathfrak{e}_i \mathfrak{e}_k = (\mathfrak{e}_i + \mathfrak{e}_k)/2$ forces $\nu_i \nu_k u_i u_k = (\nu_i^2 u_i^2 + \nu_k u_k^2)/2$ and hence

$$u_i u_k = \frac{1}{2}\left(\frac{\nu_i}{\nu_k} u_i^2 + \frac{\nu_k}{\nu_i} u_k^2\right). \tag{15}$$

In view of (14) and (15) we have

$$\begin{pmatrix} \dfrac{\alpha}{\beta} & -\dfrac{\beta}{\alpha} \\ \dfrac{\alpha}{\beta} - \dfrac{1}{2}\dfrac{\nu_i}{\nu_k} & -\dfrac{1}{2}\dfrac{\nu_k}{\nu_i} \end{pmatrix} \begin{pmatrix} u_i^2 \\ u_k^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and since $u_i^2, u_k^2 \neq 0$, the above matrix is non-regular. Since its determinant is zero, if we denote by ρ the scalar $\alpha\nu_k/\beta\nu_i$, then $0 = -\rho/2 + 1 - \rho^{-1}/2 = 1 - (\rho + \rho^{-1})/2$. This implies that $0 = 1 - 2\rho + \rho^2 = (1-\rho)^2$ and hence $\rho = 1$. Consequently, $\alpha\nu_k = \beta\nu_i$.

We now consider the case $\langle u_i, u_k \rangle^2 = (0)$. We already know, from Lemma 2.6 and Corollary 2.4, that in this case $\nu_l = 1$, $z_{l0} = 0$, $W_l = Z_l = \langle z_{lt} \colon 1 \leq t \leq r_l \rangle$, and $\mathfrak{e}_l = e + u_l \in [\Omega_l]$ for $l \in \{i, k\}$. Let $z \in W_i \cap \Psi$ and $z' \in Z_k \cap \Psi$. We will prove first that $u_i z', u_k z \in \langle u_i - u_k \rangle$ and next that $zz' \in \langle u_i - u_k \rangle$. In view of Corollary 2.4, the elements defined in (11) have the following form:

$$f_k^- = e + u_k + \varepsilon^- z', \qquad f_k^+ = e + u_k + \varepsilon^+ z', \qquad \text{with } \varepsilon^- < 0 < \varepsilon^+.$$

If $u_i z' = \lambda u = \lambda(\alpha u_i - \beta u_k)$, then we define the mapping

$$\phi \colon [f_k^-, f_k^+] \to [e, e + u_i, e + u_k]$$

$$x_\varepsilon = e + u_k + \varepsilon z' \to (e + u_i) x_\varepsilon = e + \left(\tfrac{1}{2} + \varepsilon\lambda\alpha\right) u_i + \left(\tfrac{1}{2} - \varepsilon\lambda\beta\right) u_k.$$

The mapping ϕ is well defined, because $(e + u_i)x_\varepsilon \in \Delta \cap \langle e, u_i, u_k \rangle = [e, e + u_i, e + u_k]$ by previous results. If we compute $(e + u_i)x_\varepsilon = -\varepsilon\lambda(\alpha - \beta)e + (1/2 + \varepsilon\lambda\alpha)(e + u_i) + (1/2 - \varepsilon\lambda\beta)(e + u_k) \in [e, e + u_i, e + u_k]$ we obtain that $\lambda(\alpha - \beta) = 0$, because in other cases either $-\varepsilon^+\lambda(\alpha - \beta) < 0$ or $-\varepsilon^-\lambda(\alpha - \beta) < 0$ is a contradiction.

Therefore, if $u_i z' \neq (0)$, then $(\mathbb{R}u_i)Z_k + Z_i Z_k \in \langle u_i - u_k \rangle$, since $(\mathbb{R}u_i)Z_k + Z_i Z_k \subset \langle u_i, u_k \rangle \cap U_0$. We obtain a similar result if $(\mathbb{R}u_k)Z_i \neq (0)$. Thus the lemma is proved if $(\mathbb{R}u_i)Z_k + (\mathbb{R}u_k)Z_i \neq (0)$.

Finally, we assume that $(\mathbb{R}u_i)Z_k + (\mathbb{R}u_k)Z_i = (0)$. By (11) and Corollary 2.4, for $z \in Z_i \cap \Psi$ there exist vectors f_i^-, f_i^+ in Δ with the form $f_i^- = e + u_i + \eta^- z$, $f_i^+ = e + u_i + \eta^+ z$ with $\eta^- < 0 < \eta^+$. If $zz' + \mu u = \mu(\alpha u_i - \beta u_k)$, then we consider the mapping

$$\psi \colon [f_i^-, f_i^+] \times [f_k^-, f_k^+] \to [e, e + u_i, e + u_k]$$

$$(x_\eta = e + u_i + \eta z, y_\varepsilon = e + u_k + \varepsilon z')$$

$$\to x_\eta y_\varepsilon = e + \left(\tfrac{1}{2} + \eta\varepsilon\mu\alpha\right) u_i + \left(\tfrac{1}{2} - \eta\varepsilon\mu\beta\right) u_k.$$

This mapping is well defined, and the products $x_\eta y_\varepsilon = -\eta\varepsilon\mu(\alpha-\beta)e + (1/2+\eta\varepsilon\mu\alpha)(e+u_i) + (\frac{1}{2}-\eta\varepsilon\mu\alpha)(e+u_k)$ force that $\mu(\alpha-\beta)=0$. ∎

LEMMA 2.9. *The faces* $[\mathrm{supp}(e)]$ *and* $[\mathrm{supp}(e+u_i)]$ *with* $u_i \in U_0$ *are* 0-*essential. Furthermore, if* $u_i, u_k \in U_0$ $(1 < i < k)$, *then* $\mathrm{supp}_\Psi(z_{i0})$ *and* $\mathrm{supp}_\Psi(z_{k0})$ *are disjoint subsets of* $\{z_{0_{r_0+1}}, \dots, z_{0n_0}\}$.

Proof. Let $\Phi' = \Omega_1 \cup \Omega_i$ with $i > 1$ and $u_i \in U_0$. Then $B = \langle \Phi' \rangle$ is an exceptional Bernstein subalgebra of rank 2 with Φ' as a stochastic basis. It is easy to check that $\Gamma_1 = [\mathrm{supp}(e)] = [e_{1t} : 0 \le t \le r_1]$ is a 0-essential face since $W_1^2 = (0)$. By Lemma 1.1 the face $[\Phi']$ has at least two disjoint 0-essential faces. Let Γ_i be a 0-essential face of $[\Phi']$ different from Γ_1. then $\Gamma_i^2 = \{e + \lambda u_i\}$ with $\lambda \ne 0$. Since Γ_1 and $\Gamma_i = [\mathrm{supp}(e + \lambda u_i)]$ are disjoint faces, we get that

$$z_{i0} \in \langle z_{1t} : r_1 < t \le n_1 \rangle. \tag{16}$$

Now relation (10) implies that $\lambda = 1$ since $\langle e, u_i\rangle \cap [\Phi] = \langle e, u_i \rangle \cap [\Phi'] = \langle e, u_i \rangle \cap [\Gamma_1 \cup \Gamma_i] = [\Gamma_1^2 \cup \Gamma_i^2] = [e, e + \lambda u_i]$.

Let us now assume that $u_i, u_k \in U_0$ with $1 < i < k$. Then by the first part of Lemma 2.9 we have that Γ_0, Γ_i, and $\Gamma_k = [\mathrm{supp}(e + u_k)]$ are different 0-essential faces of Δ. On the other hand, we already know that 0-essential faces are disjoint, so $\Gamma_1 \cap \Gamma_i = \Gamma_0 \cap \Gamma_k = \Gamma_i \cap \Gamma_k = \emptyset$. This implies that $z_{i0}, z_{k0} \in \langle z_{1t}\colon\ r < t \le n_1 \rangle$ and $\mathrm{supp}_\Psi(z_{i0}) \cap \mathrm{supp}_\Psi(z_{k0}) = \emptyset$. ∎

The following theorem can be proved in the same way that Theorem 2.1 of [6] was proved. The same notation will be used.

THEOREM 2.1. *For every idempotent element* $\bar{e} := e_{lt}^2$, $1 \le l \le m$, *with face* $[\mathrm{supp}(\bar{e})]$ 0-*essential, there is a basis* $\{\bar{u}_2, \dots, \bar{u}_m\}$ *of* $U_{\bar{e}}$ *and elements* $\bar{z}_i$ *in* $Z_{\bar{e}}$ *such that elements of* Φ *can be expressed as*

$$e_i = \bar{e} + \sum_{k=1}^{m-1} \overline{\mu}_{ik}\, \overline{\mu}_k + \bar{z}_i \qquad (1 \le i \le n),$$

where $\overline{\mu}_{ik} \ge 0$ *and* $\bar{e} + \lambda \bar{u}_k \in [\Phi]$ *if and only if* $0 \le \lambda \le 1$ *for all* i, k. *Furthermore, if the stochastic algebra is non-degenerate, then for every i there are at most two non-zero coefficients* $\overline{\mu}_{it}$.

Proof. We will use induction on $m = \mathrm{rk}(A)$. If $m = 1$, then the result is trivial. Let $m > 1$. If we prove that there exists Φ' a subsequence of Φ such that $e_{lt} \in \Phi'$ and the subspace $B = \langle \Phi' \rangle$ is a subalgebra of A of rank $m-1$, then by induction assumption there exists a basis $\{\bar{u}_2, \dots,$

$\bar{u}_{m-1}\}$ of $U_{\bar{e}} \cap B$ and elements $\bar{z}_i$ in $Z_{\bar{e}} \cap B$ such that

$$e_i = \bar{e} + \sum_{k=1}^{m-2} \bar{\mu}_{ik}\bar{u}_k + \bar{z}_i \qquad (e_i \in \Phi'),$$

where $\bar{\mu}_{ik} \geq 0$. Proceeding in the same way as in Theorem 2.1 of [6] we obtain the result. Thus, it remains to prove the existence of any such subsequence Φ'. If $m = 2$ then we can take as Φ' the list $\operatorname{supp}_\Phi(\bar{e})$. Finally, if $m > 2$ we take an index $k \neq l$, $2 \leq k \leq m$. Let Φ' be the set $\{e_i \in \Phi\colon \ \pi_k(e_i) = 0\}$, where $\pi_k(x)$ is the coefficient of u_k in the Ψ expression of x. It is easy to check, using previous results, that $B = \langle \Phi' \rangle$ is a subalgebra of A of rank $m - 1$, and obviously $e_{l_t} \in \Phi'$. ∎

LEMMA 2.10. *If A is non-regular, then there is a presentation of Φ of the form given by* (7)–(10) *with some $u_k \in U_0$.*

Proof. Let us suppose a presentation with no $u_i \in U_0$. Then, according to Lemma 2.5, Lemma 2.8, and non-regularity of the algebra, there exist indices i, k such that $(0) \neq (\mathbb{R}u_i \oplus Z_i)Z_k \subset \langle \nu_i u_i - \nu_k u_k \rangle \subset U_0$, and the vectors u_i^2, u_k^2 and $u_i u_k$ are linearly dependent. We can take $\bar{e} := e_{i0}^2 = e + \nu_i u_i + \nu_i^2 u_i^2 = \mathfrak{e}_i$. It is easy to prove that $[\operatorname{supp}(\bar{e})]$ is a 0-essential face since $(\mathbb{R}u_i \oplus Z_i)Z_i \subset (\mathbb{R}u_i) \cap U_0 = \{0\}$. In view of Theorem 2.1, following the ideas of [6], there is a presentation for Φ of the form given in (7)–(10) with respect to the idempotent $\bar{e}$ and the corresponding Peirce decomposition $A = \mathbb{R}\bar{e} \oplus U_{\bar{e}} \oplus Z_{\bar{e}}$. Indeed, $\overline{\Omega}_1 = \{e_i \in \Phi\colon \ \bar{e}e_i = \bar{e}\}$ and $\overline{\Omega}_1 = \{e_i \in \Phi\colon 0 \neq \bar{e}e_i - \bar{e} \in \mathbb{R}\bar{u}_i\}$ for $i = 2, \dots, m$. Next, we shall prove that there is any basic vector e_{kt} of Φ with the $U_{\bar{e}}$-component in U_0. Notice first that (6) implies that

$$\bar{u}_i := u_i + 2(\nu_i u_i)u_i = u_i + 2\nu_i u_i^2,$$

$$\bar{u}_k := u_k + 2(\nu_i u_i)u_k = u_k + 2\nu_i u_i u_k$$

$$\nu_i \bar{u}_i - \nu_j \bar{u}_k = (\nu_i u_i - \nu_k u_k) + 2(\nu_i u_i)(\nu_i u_i - \nu_k u_k) = \nu_i u_i - \nu_k u_k$$

are elements in $U_{\bar{e}}$. Let $(\nu_i u_i + \nu_i^2 u_i^2)z_{kt} := \beta_t(\nu_i u_i - \nu_k u_k)$, where $\beta_t \in \mathbb{R}$. Then

$$\bar{z}_{kt} := z_{kt} - 2\left(\nu_i u_i + \nu_i^2 u_i^2\right)z_{kt} = z_{kt} - 2\beta_t(\nu_i u_i - \nu_k u_k)$$

is an element in $Z_{\bar{e}}$ for $t = 1, \dots, r_k$. Also the vectors u_i^2, $u_i u_k$, u_k^2, z_{i0}, and z_{k0} are in $Z_{\bar{e}}$ since $(U_e \oplus U_e^2)U_e^2 = (0)$ and $z_{i0}, z_{k0} \in \langle u_i^2 \rangle$, as we proved in (13). For each t, $1 \leq t \leq r_k$, the expression of e_{kt} using $\bar{e}$ is

$$\begin{aligned} e_{kt} &= e + \nu_k u_k + z_{kt} = \bar{e} + (\nu_k u_k - \nu_i u_i) - \nu_i^2 u_i^2 + z_{kt} \\ &= \bar{e} + \big((1-2\beta_t)(\nu_k \bar{u}_k - \nu_i \bar{u}_i)\big) \\ &\quad + \big(\bar{z}_{kt} - 2(1-2\beta_t)\nu_i u_i(\nu_k u_k - \nu_i u_i) - \nu_i^2 u_i^2\big) \\ &= \bar{e} + \big((1-2\beta_t)(\nu_k \bar{u}_k - \nu_i \bar{u}_i)\big) + \big(\bar{z}_{kt} - \nu_i^2 u_i^2\big), \end{aligned}$$

and in an analogous way we obtain that the $U_{\bar{e}}$-component of $e_{k0} = e + \nu_k u_k + z_{k0} + \sum_{t=1}^{r_k} \alpha_{kt} z_{kt}$ is

$$\left(1 - 2\sum_{t=1}^{r_k} \alpha_{kt}\,\beta_t\right)(\nu_k \bar{u}_k - \nu_i \bar{u}_i),$$

since we have obtained in the proof of Lemma 2.7 that $z_{k0} \in \langle u_i^2 \rangle$. Consequently, either for any t, $1 - 2\beta_t \neq 0$ or $1 - 2\sum_{t=1}^{r_k} \alpha_{kt}\,\beta_t \neq 0$, since $\alpha_{kt} < 0$. Finally, we note that $\nu_k \bar{u}_k - \nu_i \bar{u}_i = (\nu_k u_k - \nu_i u_i) + 2\nu_i u_i(\nu_k u_k - \nu_i u_i) = \nu_k u_k - \nu_i u_i \in U_0$. ∎

Since the Bernstein problem in the regular case was studied and solved by Lyubich, we will consider from now that the non-degenerate simplicial algebra (A, Δ) is non-regular. Also we assume in this section a presentation for Φ with the element $u_h \in U_0$, $h > 1$.

LEMMA 2.11. *If $u_i \notin U_0$, then $\nu_i = 1$.*

Proof. Let $\Phi' = \Omega_0 \cup \Omega_i \cup \Omega_h$. Then $B = \langle \Phi' \rangle$ is a Bernstein subalgebra of rank 3 with Φ' as a stochastic basis. By Lemma 2.9 for each $z \in \operatorname{supp}_\Psi(B)$ the set $\{a \in \Phi' \colon z \in \operatorname{supp}_\Psi(a)\}$ has at most three elements. Thus, geometrically, it is clear that $(\mathbb{R}e \oplus U_e) \cap [\Phi'] = [e, e + u_i, e + u_h]$. If $\nu_i = \max(\nu_{it})$ then, as we saw in Lemma 2.6, $\mathfrak{e}_i = e + \nu_i u_i + \nu_i^2 u_i^2 \in [\Phi']$, and hence $\mathfrak{e}_i(e + u_h) = e + (\nu_i u_i + u_h)/2 \in \langle e, u_i, u_h \rangle \cap [\Phi'] = [e, e + u_i, e + u_h]$. Since

$$e + \frac{1}{2}(\nu_i u_i + u_h) = \frac{(1-\nu_i)}{2} e + \frac{\nu_i}{2}(e + u_i) + \frac{1}{2}(e + u_h),$$

it follows that $\nu_i \leq 1$. The reverse inequality is trivial since $e + u_i \in \Delta$. ∎

Now the following results can easily be proved.

LEMMA 2.12. (i) *If $u_i \notin U_0$, then $z_{i0} = 0$.*

(ii) *If $u_i, u_k \notin U_0$ and $u_i - u_k \in U_0$ with $i \neq k$, then $u_i^2 = (0) = u_i u_k = u_k^2$.*

(iii) *The following relation holds*:

$$(\mathbb{R}e \oplus U_e) \cap \Delta = [e + u_k \colon 1 \le k \le m].$$

Proof. We will assume that $u_i \notin U_0$. We know that $e + u_i \in [\Omega_0 \cup \{e_{it}\}_{t=0}^{r_i}]$ and hence $e + u_i = \sum_{t=0}^{n_0} \xi_t e_{0t} + \sum_{t=0}^{r_i} \xi_t e_{it}$, where $\xi_t, \zeta_t \ge 0$ and $\sum_{t=0}^{n_0} \xi_t + \sum_{t=0}^{r_i} \zeta_t = 1$. Then $1 = \pi_i(e + u_i) = \sum_{t=0}^{n_0} \xi_t \pi_i(e_{0t}) + \sum_{t=0}^{r_i} \zeta_t \pi_i(e_{it}) = \sum_{t=0}^{r_i} \zeta_t \nu_{it} = \sum_{t=0}^{r_i} \zeta_t$. Thus $\xi_t = 0$ for all t and $e + u_i \in [e_{it}\colon 0 \le t \le r_t]$, and this forces $z_{i0} = 0$.

Next we assume that $u_i, u_k \notin U_0$ and $\nu_i u_i - \nu_k u_k = u_i - u_k$ belongs to U_0. In view of (13) we have $\langle u_i^2 \rangle = \langle u_i u_k \rangle = \langle u_k^2 \rangle = \langle z_{i0} \rangle = \{0\}$. Thus, (ii) of Lemma 2.12 is proved.

Now let us assume that $x = e + u \in (\mathbb{R}e \oplus U_e) \cap \Delta$. First we note that if $a = e + u_i + z \in \operatorname{supp}(x)$, then $z \in W$, since in other case $z \in Y \cap \Psi$, and we know that $z \notin \operatorname{supp}_\Psi(a')$ for all $a' \in \Phi$, $a' \ne a$. Then $x = \mu_1 x_1 + \mu_2 x_2 + \cdots + \mu_m x_m$, where $x_i \in [\operatorname{supp}(e + u_i)]$, $\mu_i \ge 0$, and $\mu_1 + \mu_2 + \cdots + \mu_m = 1$. Also we have already proved that

$$W_i \cap W_k = \{0\} \qquad (i \ne k), \tag{17}$$

since first if $u_i, u_k \in U_0$, then $[\operatorname{supp}(e + u_i)]$ and $[\operatorname{supp}(e + u_k)]$ are disjoint 0-essential faces. Next if $u_i \in U_0$ and $u_k \notin U_0$, then $W_i \subset \langle z_{1t}\colon r_1 < t \le n_1 \rangle + \langle z_{it}\colon 1 \le t \le r_i \rangle$ and $W_k = \langle z_{kt}\colon 1 \le t \le r_k \rangle$. Finally, if $u_i, u_k \notin U_0$, then $W_i = \langle z_{it}\colon 1 \le t \le r_i \rangle$ and $W_k = \langle z_{kt}\colon 1 \le t \le r_k \rangle$. In view of relation (17), we get that $x_i \in \mathbb{R}e \oplus U_e$ whenever $\mu_i \ne 0$. Thus $x_i = e + \lambda_i u_i$ with $0 \le \lambda_i \le 1$. ∎

LEMMA 2.13. *The relations* $\lambda_{jt} + \overline{\lambda_{jt}} = 1$ *hold for all* j *and* t.

Proof. From (viii) and (iii) of Lemma 2.2 follows that

$$\Omega_{m+j} \subset \operatorname{supp}\Big((e + u_{i_j})(e + u_{k_j})\Big) = \operatorname{supp}\Big(e + \tfrac{1}{2}(u_{i_j} + u_{k_j}) + u_{i_j} u_{k_j}\Big),$$

and hence by Lemma 2.12 there exists $l \in \{1, \ldots, n_{m+j}\}$ satisfying $\lambda_{jl} + \overline{\lambda_{jl}} \ge 1$. The index l can be selected in a way such that $\lambda_{jl} + \overline{\lambda_{jl}} \ge \lambda_{jt} + \overline{\lambda_{jt}}$ for all index t. Then, as above, from (viii) and (iii) of Lemma 2.2 it follows that $\Omega_{m+j} \subset \operatorname{supp}(e_{m+jl}^2) = \operatorname{supp}(e + \lambda_{jl} u_{i_j} + \overline{\lambda_{jl}} u_{k_j} + \lambda_{jl}\overline{\lambda_{jl}} u_{i_j} u_{k_j})$, and therefore $\lambda_{jt} + \overline{\lambda_{jt}} = \lambda_{jl} + \overline{\lambda_{jl}} \ge 1$ for all t.

To check the reverse inequality, we note that A^2 is regular and hence

$$(e + u_h) e_{m+jt}^2 = e + \tfrac{1}{2}\Big(u_h + \lambda_{jt} u_{i_j} + \overline{\lambda_{jt}} u_{k_j}\Big) \in \Delta.$$

This, together with Lemma 2.12, implies that $(1 + \lambda_{jt} + \overline{\lambda_{jt}})/2 \le 1$, that is, $\lambda_{jt} + \overline{\lambda_{jt}} \le 1$. ∎

LEMMA 2.14. *If* $u_i \notin U_0$, *then* $\nu_{it} = 1$ *for all* t.

Proof. Let $a = e + \nu u_i + z \in \Phi$. If $z = 0$, then $e + u_i \in \Delta$, $e + \nu u_i \in \Phi$ imply that $\nu = 1$. If $z \in W_i$ then $\nu = 1$ by Lemma 2.11. Finally, we shall assume that $z \in Y_i$. Then there exists k with $z \in \operatorname{supp}_\Psi(u_i u_k)$. In view of (iii) of Lemma 2.2,

$$a \in \operatorname{supp}\big((e + u_i)(e + u_k)\big) = \operatorname{supp}\big(e + \tfrac{1}{2}(u_i + u_k) + u_i u_k\big).$$

This, together with the previous lemmas, implies $\nu = 1$. ∎

COROLLARY 2.5. *The relation* $Y_1 = \{0\}$ *holds.*

3. THE MAIN THEOREM

As suggested by the previous results, we will introduce a new notation for Φ and Ψ. We recall that u_1 denotes the null vector. Thus, for (A, Δ), a non-degenerate non-regular simplicial Bernstein algebra of type (m, δ), $n = m + \delta$, we have

(a) an idempotent $e \in \Delta$, a basis $u_2, \dots, u_m$ of U_e, and a basis $\{v_{lt_l}\}$ of Z_e,

$$\Psi = \big\{e, u_i, v_{lt_l} \colon 2 \le i \le m, 1 \le l \le m + \gamma, 0 \le t_l \le d_l\big\},$$

(b) a partition $\{I_0, I_1, I_2\}$ of the set $\{1, \dots, m\}$, where $i \in I_0$ if and only if $u_i \in U_0$, $i \in I_1$ if and only if $i \notin I_0$, and there exists an index k, $k \ne i$ such that $u_i - u_k \in U_0$,

(c) the expression of $\{\Phi_l\}_{l=1}^{m+\gamma}$, after reindexing, with respect to Ψ is of the form

$$(i \in I_0),\ \Phi_i = \begin{cases} a_{i0} = e + \rho_{i0} u_i + \sum_{k=0}^{s_i} \beta_{ik} v_{ik} & \\ a_{it} = e + \rho_{it} u_i + v_{it} & (1 \le t \le s_i = d_i) \end{cases} \tag{18}$$

$$(i \in I_1),\ \Phi_i = \begin{cases} a_{i0} = e + u_i + \sum_{k=1}^{s_i} \beta_{ik} v_{ik} & \\ a_{it} = e + u_i + v_{it} & (1 \le t \le s_i = d_i) \end{cases} \tag{19}$$

$$(i \in I_2),\ \Phi_i = \begin{cases} a_{i0} = e + u_i + \sum_{k=1}^{s_i} \beta_{ik} v_{ik} & \\ a_{it} = e + u_i + v_{it} & (1 \le t \le d_i) \end{cases} \tag{20}$$

$$\Phi_{m+j} = \Big\{ a_{m+jt} = e + \lambda_{jt} u_{i_j} + \overline{\lambda_{jt}} u_{k_j} v_{m+jt} \qquad (1 \le t \le d_{m+j}), \tag{21}$$

where $0 \le \rho_{it} \le 2$, $\beta_{it} < 0$, $0 < \lambda_{jt}, \overline{\lambda_{jt}}$, and $\lambda_{jt} + \overline{\lambda_{jt}} = 1$. All the pairs (i_j, k_j) are distinct, and $2 \le i_j < k_j \le m$ with $i_j, k_j \in I_2$. For each $i \in I_0 \cup I_1$ the face $\Lambda_i := [\Phi_i]$ is 0-essential and

$$\Lambda_i^2 = \{e + u_i\}.$$

For each $i \in I_2$ the face $\Lambda_i := [\Phi_i]$ is invariant, that is, $\Lambda_i \cdot \Lambda_i \subset \Lambda_i$, and it satisfies

$$e + u_i, e + u_i + u_i^2 \in \Lambda_i.$$

The relationship between the two notations is (agree $s_{m+j} := 0$)

$$\begin{aligned} W_i &= \langle v_{it} : 1 \le t \le s_i \rangle, \qquad Y_i = \langle v_{it} : s_i < t \le d_i \rangle \\ \Phi_1 &= \operatorname{supp}(e) = \{e_{1t} : 0 \le t \le r_1\} \\ \Phi_i &= \operatorname{supp}(e + u_i) \cup \Omega_i \qquad (\text{if } 1 \ne i \in I_0) \\ \Phi_i &= \Omega_i \qquad (\text{if } i \in I_1 \cup I_2) \\ \Phi_{m+j} &= \Omega_{m+j}. \end{aligned}$$

LEMMA 3.1. *The relations*

$$\Lambda_i \cdot (e + u_k) = \left\{e + \tfrac{1}{2}(u_i + u_k)\right\}, \qquad \Lambda_i \cdot W_k = (0)$$

hold for all $i \in I_0$ *and all* $k \in I_1 \cup I_2$.

Proof. First we shall prove that $\Lambda_i \cdot (e + u_k) = \{e + (u_i + u_k)/2\}$. Let $a = e + \rho u_i + v \in \Phi_i$. In view of the fact that Λ_i is 0-essential, there exists $b \in \Lambda_i$ such that $e + u_i = (1 - \alpha)a + \alpha b$, where $0 < \alpha < 1$. Since $\Lambda_i(e + u_k) \subset \langle e, u_i, u_k \rangle \cap \Delta = [e, e + u_i, e + u_k]$ we have

$$\begin{aligned} \tfrac{1}{2}(e + u_i) + \tfrac{1}{2}(e + u_k) &= (e + u_i)(e + u_k) \in \operatorname{Int}\{[a, b] \cdot (e + u_k)\} \\ &= \operatorname{Int}[a(e + u_k), b(e + u_k)] \\ &\subset [e, e + u_i, e + u_k]. \end{aligned}$$

This implies that $[a(e + u_k), b(e + u_k)] \subset [e + u_i, e + u_k]$. On the other hand, we know that $vu_k \in \mathbb{R}u_i$ and hence $vu_k = \theta u_i$, where θ is a scalar, so $a(e + u_k)$ is of the form $e + (\rho/2 + \theta)u_i + (1/2)u_k$. Then, because this element belongs to $[e + u_i, e + u_k]$, we get that $\rho/2 + \theta = 1/2$.

Finally, we shall show that $\Lambda_i \cdot W_k = (0)$. Let $v' \in W_k \cap \Psi$. We consider the elements in Δ already defined, $f^- = e + u_k + \varepsilon^- v'$ and $f^+ = e + u_k + \varepsilon^+ v'$. We know that $(\mathbb{R}u_i \oplus Z_i)(\mathbb{R}v') \subset \mathbb{R}u_i$, so that for each $x \in [\operatorname{supp}(e$

$+ u_i)]$ we have $xv' = \vartheta u_i$, with $\vartheta \in \mathbb{R}$ depending on x. Then by the first part we have

$$xf^\tau = x(e + u_k) + \varepsilon^\tau xv' = e + \left(\tfrac{1}{2} + \varepsilon^\tau\vartheta\right)u_i + \tfrac{1}{2}u_k \in \Delta, \qquad \tau \in \{-, +\}.$$

But now (iii) of Corollary 2.12 implies that $\vartheta = 0$. In particular, for $x = e + u_i$ we have that $u_i v' = 0$, and next for $x = a$ we have $0 = av' = \rho u_i v' + vv' = vv'$. Therefore we have proved that $\Lambda_i W_k = \{0\}$. ∎

COROLLARY 3.1. *For all $i \in I_0$ and all $k \in I_1 \cup I_2$, we have*

$$(\mathbb{R}u_i) \cdot W_k = (0), \qquad Z_i \cdot W_k = (0).$$

LEMMA 3.2. *The following relation holds*:

$$U_e \cdot \left\langle \operatorname{supp}_\Psi\left(U_e^2\right)\right\rangle = (0).$$

Proof. Let $u_k, u_l \notin U_0$ with $k \leq l$ and $u_k u_l \neq 0$. Again arguing as at beginning of the proof of Lemma 2.4 of [6], for every $v \in \operatorname{supp}_\Psi(u_k u_l)$ there is an element $f_v = e + u_v + \gamma_v v \in \Delta$ where u_v belongs to $\langle u_k, u_l\rangle$, and γ_v and the coefficient of v in the expression of $u_k u_l$ with respect to the basis Ψ have the same sign. Indeed, if $v \notin W$ we can take as f_v a basic vector in Φ, and for $v \in W$ we can use (iv) of Lemma 2.2. If $v \in Z_k$ then $u_v = u_k$, and analogously if $v \in Z_l$ then $u_v = u_l$. Finally, if $v = v_{m+jt} \in Z_{m+j}$, then $(k, l) = (i_j, k_j)$ and $u_v = \lambda_{jt} u_k + \overline{\lambda_{jt}} u_l$. Thus, in all the cases $u_v = \alpha_v u_k + (1 - \alpha_v)u_l$, where $0 \leq \alpha_v \leq 1$. There exists scalars $\zeta_v > 0$ verifying $\sum\{\zeta_v\colon v \in \operatorname{supp}_\Psi(u_k u_l)\} = 1$ and

$$0 \neq \sum\{\zeta_v \gamma_v v\colon v \in \operatorname{supp}_\Psi(u_k u_l)\} \in \langle u_k u_l\rangle. \tag{22}$$

Let $u_i \in U_e$. If $u_i \notin U_0$, then $(\mathbb{R}u_i)\langle \operatorname{supp}_\Psi(U_e^2)\rangle = (0)$, since we know that

$$(\mathbb{R}u_i)\left\langle \operatorname{supp}_\Psi\left(U_e^2\right)\right\rangle \subset \mathbb{R}u_i \cap U_0.$$

In another case $u_i \in U_0$ and $u_i v = \vartheta_v u_i$, where $\vartheta_v \in \mathbb{R}$. Then the products

$$\begin{aligned}
(e + u_i)f_v &= (e + u_i)(e + \alpha_v u_k + (1 - \alpha_v)u_l) + \gamma_v(e + u_i)v \\
&= \alpha_v(e + u_i)(e + u_k) + (1 - \alpha_v)(e + u_i)(e + u_l) + \gamma_v u_i v \\
&= \alpha_v\left(e + \tfrac{1}{2}(u_i + u_k)\right) + (1 - \alpha_v)\left(e + \tfrac{1}{2}(u_i + u_l)\right) + \gamma_v \vartheta_v u_i \\
&= e + \left(\tfrac{1}{2} + \vartheta_v \gamma_v\right)u_i + \tfrac{1}{2}\alpha_v u_k + \tfrac{1}{2}(1 - \alpha_v)u_l
\end{aligned}$$

are in $\langle e, u_i, u_k, u_l \rangle \cap \Delta = [e, e + u_i, e + u_k, e + u_l]$, and therefore we get that

$$\vartheta_v \gamma_v \leq 0. \tag{23}$$

Finally, because $u_i \in U_0$, from relation (22) we obtain

$$\begin{aligned} 0 = u_i(u_k u_l) &= u_i\left(\sum_v \zeta_v \gamma_v v\right) = \sum_v \zeta_v \gamma_v u_i v = \sum_v \zeta_v \vartheta_v \gamma_v u_i \\ &= \left(\sum_v \zeta_v \vartheta_v \gamma_v\right) u_i, \end{aligned}$$

so this equality joint to (23) forces $\vartheta_v \gamma_v = 0$ and hence $\vartheta_v = 0$. ∎

COROLLARY 3.2. *For all $i \in I_0$, $k \in I_1 \cup I_2$, and $j \in \{1, \ldots, \gamma\}$, we have*

$$(\mathbb{R}u_i) \cdot Z_k = (0), \qquad (\mathbb{R}u_i) \cdot Z_{m+j} = (0).$$

LEMMA 3.3. *The relation*

$$Z_i \cdot Z_k = (0), \qquad Z_i \cdot Z_{m+j} = (0)$$

holds for all $i \in I_0$, $k \in I_1 \cup I_2$, and $j \in \{1, \ldots, \gamma\}$.

Proof. This part can be proved in the same way that the first part of Lemma 3.1 was proved. The same notation will be used. Consider $a = e + u_i + v$ and $a' = e + u_k + v'$ arbitrary elements in Φ_i and Φ_k, respectively. Let $b \in [\operatorname{supp}(e + u_i)]$ such that $e + u_i \in \operatorname{Int}[a, b]$ and consider $\alpha > 0$ such that $\alpha a + (1 - \alpha) b = e + u_i$. Since $vv' \in \mathbb{R}u_i$, there exists $\vartheta \in \mathbb{R}$ verifying $vv' = \vartheta u_i$. Then by the second part of Lemma 3.1 and Lemma 3.2, we have

$$\begin{aligned} \tfrac{1}{2}(e + u_i) + \tfrac{1}{2}(e + u_k) = (e + u_i)a' &\in \operatorname{Int}\{[a, b] \cdot a'\} \\ &= \operatorname{Int}[aa', ba'] \subset [e, e + u_i, e + u_k], \end{aligned}$$

and hence $[aa', ba'] \subset [e + u_i, e + u_k]$. On the other hand, we know that $aa' = a(e + u_k) + av' = a(e + u_k) + u_i v' + vv' = a(e + u_k) + vv' = (e + (1/2)u_i + (1/2)u_k) + (\vartheta u_i) = e + (1/2 + \vartheta)u_i + (1/2)u_k$. Then, because this element belongs to $[e + u_i, e + u_k]$, we have that $\vartheta = 0$. This forces $vv' = 0$.

Finally, consider $a'' = e + \lambda u_{i_j} + \bar{\lambda} u_{k_j} + v'' \in \Phi_{m+j}$. Obviously, $\lambda + \bar{\lambda} = 1$. We know that $vv'' \in \mathbb{R}u_i$, so $vv'' = \theta u_i$ with $\theta \in \mathbb{R}$. Then

$$\begin{aligned}\frac{1}{2}(e + u_i) + \frac{\lambda}{2}(e + u_{i_j}) + \frac{\bar{\lambda}}{2}(e + u_{k_j}) \\ &= (e + u_i)a'' \in \operatorname{Int}\{[a, b] \cdot a''\} \\ &= \operatorname{Int}[aa'', ba''] \subset [e, e + u_i, e + u_{i_j}, e + u_{k_j}],\end{aligned}$$

and hence $[aa'', ba''] \subset [e + u_i, e + u_{i_j}, e + u_{k_j}]$. Next we know that

$$\begin{aligned}aa'' &= a\left(e + \lambda u_{i_j} + \bar{\lambda} u_{k_j}\right) + av'' \\ &= \lambda a(e + u_{i_j}) + \bar{\lambda} a(e + u_{k_j}) + u_i v'' + vv'' \\ &= \lambda\left(e + \frac{1}{2}u_i + \frac{1}{2}u_{i_j}\right) + \bar{\lambda}\left(e + \frac{1}{2}u_i + \frac{1}{2}u_{k_j}\right) + vv'' \\ &= e + \frac{1}{2}u_i + \frac{\lambda}{2}u_{i_j} + \frac{\bar{\lambda}}{2}u_{k_j} + \vartheta u_i \\ &= e + \left(\frac{1}{2} + \vartheta\right)u_i + \frac{\lambda}{2}u_{i_j} + \frac{\bar{\lambda}}{2}u_{k_j}.\end{aligned}$$

Then, because this element belongs to $[e + u_i, e + u_{i_j}, e + u_{k_j}]$ we have that $\theta = 0$. This forces $vv'' = 0$. ∎

Lemmas 3.1, 3.2, and 3.3 prove the following.

LEMMA 3.4. *The relations*

$$\Lambda_i \cdot \Lambda_k = \left\{e + \tfrac{1}{2}(u_i + u_k)\right\},$$

$$\Lambda_i \cdot \{a_{m+jt}\} = \left\{e + \tfrac{1}{2}\left(u_i + \lambda_{jt} u_{i_j} + \overline{\lambda_{jt}} u_{k_j}\right)\right\}$$

hold where $i \in I_0$, $k \in I_1 \cup I_2$, *and* $j \in \{1, 2, \ldots, \gamma\}$.

We may now prove the following lemma.

LEMMA 3.5. *The following relations hold*:

$$Y = \left(\sum_{k \in I_2} Y_k\right) \oplus \left(\sum_{j=1}^{\gamma} Y_{m+j}\right) \subset \left\langle \operatorname{supp}_{\Omega}\left(U_e^2\right)\right\rangle \subset \left(\sum_{k \in I_2} Z_k\right) \oplus \left(\sum_{j=1}^{\gamma} Y_{m+j}\right)$$
$$\subset \operatorname{ann}(A).$$

Proof. We shall prove the last inclusion. Let $k \in I_2$. In view of Lemmas 3.1, 3.2, 3.3, and 3.4 we have that $(\mathbb{R}u_i \oplus Z_i)(Z_k \oplus Z_{m+j}) = (0)$ for $i \in I_0$. Next, if $i \in I_1 \cap I_2$ then we know that $(\mathbb{R}u_i \oplus Z_i)(Z_k \oplus Z_{m+j}) \subset \langle u_i, u_k \rangle \cap U_0$, but definitions of I_1 and I_2 force the relation $\langle u_i, u_k \rangle \cap U_0 = \{0\}$. ∎

COROLLARY 3.3. *The following relation holds*: $Z_e \cdot \langle \operatorname{supp}(U_e^2) \rangle = (0)$.

LEMMA 3.6. *Every non-degenerate and externally irreducible simplicial Bernstein algebra is regular.*

Proof. Let A be a non-degenerate non-regular simplicial Bernstein algebra. In view of previous results, there is $k \in I_0 \cup I_1$ such that $W_k \neq (0)$. If $x = \sum_{i,t} x_{it} a_{it}$ and $x^2 = \sum_{it} x'_{it} a_{it}$, then $x'_{k0} = -\beta_{kt} x'_{kt}$ for $t = 1, \ldots, d_k$. Therefore the simplicial algebra is externally reducible. ∎

The above lemma proves Conjecture 5.7.15 in [18] in an affirmative way. For that we only need to remember that a normal simplicial Bernstein algebra is non-degenerate and externally irreducible. Thus we have proved: *A normal simplicial Bernstein algebra is regular.*

LEMMA 3.7. *The relation*

$$\Lambda_i \cdot \Lambda_k \in [e + u_i, e + u_k]$$

holds for all $i, k \in I_0 \cup I_1$.

Proof. Let $a \in \Phi_i$ and $a' \in \Phi_k$. Then there is $b \in [\operatorname{supp}(e + u_i)]$ and $b' \in [\operatorname{supp}(e + u_k)]$ such that $e + u_i \in \operatorname{Int}[a, b]$ and $e + u_k \in \operatorname{Int}[a', b']$. Therefore

$$\begin{aligned} \tfrac{1}{2}(e + u_i) + \tfrac{1}{2}(e + u_k) &= (e + u_i)(e + u_k) \in \operatorname{int}\{[a, b] \cdot [a', b']\} \\ &= \operatorname{Int}\{[aa', ab'] \times [ba', bb']\} \\ &\subset [e, e + u_i, e + u_k]. \end{aligned}$$

Consequently, $aa' \in [e + u_i, e + u_k]$. ∎

Now, the main theorem can easily be proved.

THEOREM 3.1. *Every non-degenerate non-regular simplicial Bernstein algebra* (A, Δ) *of rank* m *is as follows.*

Up to partition $\{I_0, I_1, I_2\}$ *of the set* $\{1, 2, \ldots, m\}$ *and to enumeration of the canonical basis* $\Phi = \{\Phi_i\}_{i=1}^{m+\gamma}$, *where* $\Phi_i = \{a_{it}\}_{t=0}^{d_i}$, *the faces* $\Lambda_i = [\Phi_i]$ *are invariant with rank equal to* 1, *that is,* $\Lambda_i^2 = \{\mathfrak{e}_i\}$, *where*

$$\mathfrak{e}_i = \sum_{t=0}^{d_i} \xi_{it} a_{it}, \qquad \xi_{it} \geq 0, \qquad \sum_{t=0}^{d_i} \xi_{it} = 1. \tag{24}$$

Furthermore, for $i \in I_0 \cup I_1$ *the face* Λ_i *is* 0*-essential, and hence* $\xi_{it} >$ *for all* t. *Their products are*

I. *For either* $i, k \in I_0$ *or* $i, k \in I_1$,

$$a_{it} a_{kl} = \zeta_{ik,tl} \mathfrak{e}_i + \zeta_{ki,lt} \mathfrak{e}_k \in [\mathfrak{e}_i, \mathfrak{e}_k], \tag{25}$$

where

$$\sum_{t,l=1}^{d_i, d_k} \xi_{it} \xi_{kl} \zeta_{ik,tl} = \sum_{t,l=1}^{d_i, d_k} \xi_{it} \xi_{kl} \zeta_{ki,lt}.$$

This means that $\mathfrak{e}_k \mathfrak{e}_k = (\mathfrak{e}_i + \mathfrak{e}_k)/2$.

II. *For* $i \in I_0$ *and* $k \in I_1$,

$$\Lambda_i \cdot \Lambda_k = \frac{\mathfrak{e}_i + \mathfrak{e}_k}{2}. \tag{26}$$

III. *For* $i \in I_0$ *and* $k \in I_2$,

$$\Lambda_i \cdot \Lambda_k = \frac{\mathfrak{e}_i + \mathfrak{f}_k}{2}, \tag{27}$$

where

$$\mathfrak{f}_k = \sum_{t=1}^{d_k} \zeta_{kt} a_{kt}, \qquad \zeta_{kt} \geq 0, \qquad \sum_{t=0}^{d_k} \zeta_{kt} = 1.$$

IV. *For each* $i \in I_1$ *and* $k \in I_2$,

$$\Lambda_i \cdot \Lambda_k = \frac{\mathfrak{e}_i + \mathfrak{f}_{ki}}{2}, \tag{28}$$

where

$$\mathfrak{f}_{ki} = \sum_{t=0}^{d_k} \zeta_{ki,t} a_{kt}, \qquad \zeta_{ki,t} \geq 0, \qquad \sum_{t=0}^{d_k} \zeta_{ki,t} = 1.$$

Besides, for each pair (i, p), $i, p \in I_1$, *such that* $\Lambda_i \cdot \Lambda_p \neq \{(\mathfrak{e}_i + \mathfrak{e}_p)/2\}$, *we have* $\mathfrak{f}_{ki} = \mathfrak{f}_{kp}$.

V. *For some distinct pairs* (i_j, k_j), $1 \leq j \leq \gamma$, *with* $i_j < k_j$ *and* $i_j, k_j \in I_2$, *we have*

$$\Lambda_{i_j} \cdot \Lambda_{k_j} = \left(\sum_{t=0}^{d_{i_j}} \alpha_{jt} a_{i_j t} \right) + \left(\sum_{t=0}^{d_{k_j}} \beta_{jt} a_{k_j t} \right) + \left(\sum_{t=0}^{d_{m+j}} \delta_{jt} a_{m+jt} \right), \tag{29}$$

where $\alpha_{jt}, \beta_{jt} \geq 0$, $\delta_{jt} > 0$, *and*

$$\left(\sum_{t=1}^{d_{i_k}} \alpha_{jt}\right) + \left(\sum_{t=1}^{d_{k_j}} \beta_{jt}\right) + \left(\sum_{t=1}^{d_{m+j}} \delta_{jt}\right) = 1.$$

For all remaining pairs (i,k) *with* $i \leq k$, $i, k \in I_2$, *we have*

$$\Lambda_i \cdot \Lambda_k = \left(\sum_{t=0}^{d_i} \alpha_{ik,t} a_{it}\right) + \left(\sum_{t=0}^{d_k} \beta_{ki,t} a_{kt}\right), \tag{30}$$

where $\alpha_{ik,t}, \beta_{ki,t} \geq 0$ *and*

$$\left(\sum_{t=0}^{d_i} \alpha_{ik,t}\right) = \tfrac{1}{2} = \left(\sum_{t=0}^{d_k} \beta_{ki,t}\right).$$

VI. *Furthermore, for* $1 \leq i \leq m$ *and* $1 \leq j \leq \gamma$,

$$a_{il} a_{m+j,t} = \lambda_{jt} a_{il} \mathfrak{e}_{i_j} + \overline{\lambda_{jt}} a_{il} \mathfrak{e}_{k_j}, \tag{31}$$

where $0 < \lambda_{jt} < 1$, $\lambda_{jt} + \overline{\lambda_{jt}} = 1$, *and*

$$\left(\sum_{t=0}^{d_{i_k}} \alpha_{jt}\right) + \left(\sum_{t=0}^{d_{m+j}} \delta_{jt} \lambda_{jt}\right) = \tfrac{1}{2} = \left(\sum_{t=0}^{d_{k_j}} \beta_{jt}\right) + \left(\sum_{t=0}^{d_{m+j}} \delta_{jt} \overline{\lambda_{jt}}\right).$$

Finally,

$$\begin{aligned} a_{m+j,l} a_{m+g,h} = {} & \lambda_{jl} \lambda_{gh} \mathfrak{e}_{i_j} \mathfrak{e}_{i_g} + \lambda_{jl} \overline{\lambda_{gh}} \mathfrak{e}_{i_j} \mathfrak{e}_{k_g} + \overline{\lambda_{jl}} \lambda_{gh} \mathfrak{e}_{k_j} \mathfrak{e}_{i_g} \\ & + \overline{\lambda_{jl} \lambda_{gh}} \mathfrak{e}_{k_j} \mathfrak{e}_{k_g}. \end{aligned} \tag{32}$$

Conversely, the above described algebra is simplicial and Bernstein.

Notice that from Theorem 3.1 we can describe explicitly all non-regular simplicial Bernstein algebras. If (A, Δ) is a simplicial Bernstein algebra, then the canonical basis has a partition $\{\Phi', \Phi''\}$ where $\Phi' = \operatorname{supp}(A^2)$. Then $B = \langle \Phi' \rangle$ is a Bernstein subalgebra of A and $(B, [\Phi'])$ is a non-degenerate stochastic Bernstein algebra. Furthermore, $\Phi \cdot \Phi \subset [\Phi']$.

From this theorem and results obtained by Lyubich in the regular case we have

COROLLARY 3.4. *Let* (A, Δ) *be a simplicial Bernstein algebra of rank* m. *Then it has at least* $\binom{m}{k}$ *non-degenerate simplicial subalgebras of rank* k *for* $k = 1, 2, \ldots, m$.

REFERENCES

1. S. N. Bernstein, Mathematical problems in modern biology, *Science Ukraine* **1** (1922), 14–19 (in Russian).
2. S. N. Bernstein, Démonstration mathématique de la loi d'hérédité de Mendel, *C. R. Acad. Sci. Paris* **177** (1923), 528–531.
3. S. N. Bernstein, Principle de stationarité et généralisation de la loi de Mendel, *C. R. Acad. Sci. Paris* **177** (1923), 581–584.
4. S. González and C. Martínez, Idempotent elements in a Bernstein algebra, *J. London Math. Soc.* (2) **42** (1991), 430–436.
5. S. González, J. Carlos Gutiérrez, and C. Martínez, The Bernstein problem in dimension 5, *J. Algebra* **177** (1995), 676–697.
6. J. Carlos Gutiérrez and C. Martínez, Nuclear Bernstein algebras with stochastic basis, *J. Algebra* **217** (1999), 300–311.
7. J. Carlos Gutiérrez, The Bernstein problem for the type $(n-2,2)$, *J. Algebra* **181** (1996), 613–627.
8. J. Carlos Gutiérrez, The Bernstein problem in dimension 6, *J. Algebra* **185** (1996), 420–439.
9. J. Carlos Gutiérrez, Structure of Bernstein population of type $(3, n-3)$, *Linear Algebra Appl.* **268** (1998), 17–32.
10. P. Holgate, Genetic algebras satisfying Bernstein's stationarity principle, *J. London Math. Soc.* (2) **9** (1975), 621–624.
11. Yu. I. Lyubich, Basis concepts and theorems of evolutionary genetics for free populations, *Russian Math. Surveys* **26**, No. 5 (1971), 51–123.
12. Yu. I. Lyubich, On the mathematical theory of hereditary, *Soviet Math. Dokl.* **14** (1973), 51–123 (in Russian).
13. Yu. I. Lyubich, Two-level Bernstein populations, *Math. USSR Sb.* **24**, No. 1 (1974), 593–615.
14. Yu. I. Lyubich, Quasilinear Bernstein populations, *Teor. Funktsiĭ Funktsional. Anal. i Prilozhen.* **26** (1976), 79–84 (in Russian).
15. Yu. I. Lyubich, Proper Bernstein populations, *Probl. Inform. Transmiss.* Jan. (1978), 228–235.
16. Yu. I. Lyubich, Bernstein algebras, *Uspekhi Mat. Nauk* **32**, No. 6 (1977), 261–263 (in Russian).
17. Yu. I. Lyubich, A topological approach to a problem in mathematical genetics, *Russian Math. Surveys* **34**, No. 6 (1979), 60–66.
18. Yu. I. Lyubich, Mathematical structures in population genetics, *in* "Biomathematics," Vol. 22, Springer-Verlag, Berlin/Heidelberg, 1992.
19. Yu. I. Lyubich, "A New Advance in the Bernstein Problem in Mathematical Genetics," Preprints, Vol. 9, Institute for Mathematical Science, Stony Brook, 1996.
20. M. Lynn Reed, Algebraic structures of genetic inheritance, *Bull. Amer. Math. Soc.* (*N.S.*) **34**, No. 2 (1997), 107–131.
21. S. Walcher, Bernstein algebras which are Jordan algebras, *Arch. Math.* **50** (1988), 218–222.